ISRAEL JOURNAL OF MATHEMATICS 154 (2006), 247–251

# THE CHERN CLASSES OF THE EIGENBUNDLES OF AN AUTOMORPHISM OF A CURVE

BY

Israel Moreno Mejía*

*Department of Mathematics and Computer Science, Bar Ilan University*
*52900 Ramat Gan, Israel*
*e-mail: morenoi@macs.biu.ac.il*

ABSTRACT

Let $X$ be smooth complex projective curve. Let $h$ be an automorphism of $X$ of order $p$. We improve a formula to compute the characteristic classes of the normal bundles of certain components of the fixed point set of $h$ acting on the symmetric products of $X$.

## 1. Introduction

We consider the following situation. Let $f\colon X \to Y$ be a finite morphism of degree $p$ between two smooth complex projective curves. Let $S^kX$ denote the $k$-symmetric product of $X$. There is a morphism $i\colon S^kY \to S^{pk}X$ defined by $i(D) = f^*D$. If $Y$ is the quotient $X/\langle h\rangle$, where $h$ is an automorphism of $X$, and if $f$ is the quotient map, then $S^kY$ can be identified with a component of the fixed point set of $h$ acting on $S^{pk}X$. The normal bundle $N_i$ of this component has a decomposition into eigenbundles $N_i = \bigoplus_{j=1}^{p-1} N_i(\nu^j)$. The problem is to compute the characteristic classes $\mathcal{U}_j(N_i(\nu^j))$ which are required to apply the Holomorphic Lefschetz Theorem; see [2]. Assuming that all the the nontrivial powers of $h$ belong to the same conjugacy class of $Aut(X)$, an expression for $\mathcal{U}_j(N_i(\nu^j))$ was obtained in [4]; this was the key for the calculations in [5]. In this work we present an improved version of that formula. Consider the decomposition of $f_*\mathcal{O}_X$ into eigenbundles $f_*\mathcal{O}_X = \bigoplus_{j=0}^{p-1} \mathcal{L}_j$. The eigenbundles

* The author was supported by the Emmy Noether Research Institute for Mathematics.
Received December 29, 2004

$N_i(\nu^j)$ and $\mathcal{L}_j$ are related by means of an identity that allows us to use the Grothendieck–Riemann–Roch theorem to compute the Chern classes of $N_i(\nu^j)$, which is what we need to compute the characeristic classes of the eigenbundles $N_i(\nu^j)$.

## 2. The universal divisor

The universal effective divisor of degree $d$ on $X$ is the divisor

$$\Delta \subset X \times S^d X$$

that, for any $D \in S^d X$, cuts on $X \cong X \times \{D\}$ exactly the divisor $D$; we refer to [1] chapter IV, section 2 for more details.

PROPOSITION 2.1: *Let $\Delta_Y \subset Y \times S^k Y$ be the universal divisor and let $\pi_Y\colon \Delta_Y \to Y$ and $\pi_{S^kY}\colon \Delta_Y \to S^k Y$ be the corresponding projections. Then*

$$i^* T_{S^{pk}X} \cong \pi_{S^kY*}(\mathcal{O}_{\Delta_Y}(\Delta_Y) \otimes \pi_Y^* f_* \mathcal{O}_X).$$

*Proof:* Let $\Delta_X \subset X \times S^{pk}X$ be the universal divisor. Consider the divisor $\Delta' = (id_X \times i)^* \Delta_X \subset X \times S^k Y$ with its corresponding projections $\rho_X$ and $\rho_{S^kY}$. Then from [1], chapter IV, section 2 (Lemma 2.3 and p. 174), we have $i^* T_{S^{pk}X} \cong \rho_{S^kY*}(\mathcal{O}_{\Delta'}(\Delta'))$. Notice that $\Delta' = (f \times id_{S^kY})^* \Delta_Y$; see the diagram (Figure 1).

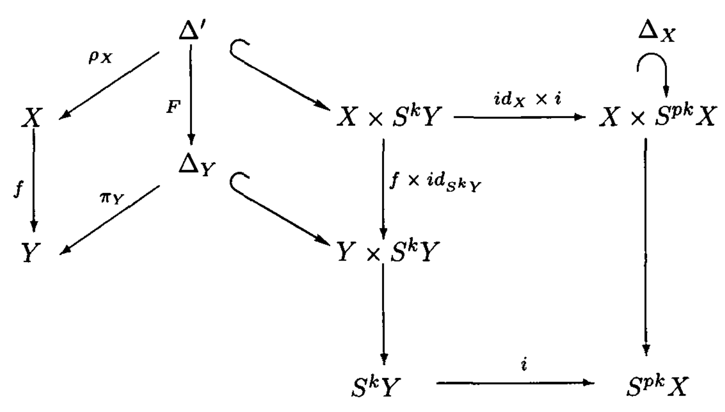


Figure 1.

Let $F$ denote $(f \times id_{S^kY})\,|_{\Delta'}$. So

$$i^*T_{S^{pk}X} \cong \pi_{S^kY*}(\mathcal{O}_{\Delta_Y}(\Delta_Y) \otimes F_*\mathcal{O}_{\Delta'}).$$

On the other hand, since $f$ is a finite morphism and $\Delta' = X \times_Y \Delta_Y$, we have $F_*\mathcal{O}_{\Delta'} = \pi_Y^* f_*\mathcal{O}_X$. ∎

If $Y = X/\langle h \rangle$ and $f$ is the quotient map, we have a decomposition

$$f_*\mathcal{O}_X = \bigoplus_{j=0}^{p-1} \mathcal{L}_j$$

where $\mathcal{L}_j$ is the subline bundle of $f_*\mathcal{O}_X$ on which the action of $h$ is multiplication by the scalar $\nu^j$, $\nu = e^{2i\pi/p}$. Therefore

$$i^*T_{S^{pk}X} \cong \bigoplus_{j=0}^{p-1} \pi_{S^kY*}(\mathcal{O}_{\Delta_Y}(\Delta_Y) \otimes \pi_Y^*\mathcal{L}_j).$$

Now we can use the Grothendieck–Riemann–Roch theorem to compute the chern classes of the vector bundles

$$i^*T_{S^{pk}X}(\nu^j) = \pi_{S^kY*}(\mathcal{O}_{\Delta_Y}(\Delta_Y) \otimes \pi_Y^*\mathcal{L}_j).$$

Let $x, \theta \in H^2(S^kY, \mathbb{Z})$ denote respectively the class of the divisor $q + S^{k-1}Y \subset S^kY$ and the pull-back of the class $\theta \in H^2(J_Y, \mathbb{Z})$ to $S^kY$, where $J_Y$ is the Jacobian variety of $Y$. With the notation of [3], $\theta = \sum_{i=1}^{g_Y} \sigma_i$, where $\sigma_i^2 = 0$ (see (5.4) in [3]); then one can write

$$e^{\alpha\theta} = \prod_{i=1}^{g_Y}(1 + \alpha\sigma_i). \tag{1}$$

LEMMA 2.2: *Let $n_j$ be the degree of $\mathcal{L}_j$. Then*

$$c(i^*T_{S^{pk}X}(\nu^j)) = (1+x)^{k+n_j+1-g_Y} e^{-\theta/(1+x)}.$$

*Proof:* The proof is essentially the same as that of Lemma 2.5 in [1], chapter VIII, section 2. Let $\pi_1 \colon Y \times S^kY \to Y$ and $\pi_2 \colon Y \times S^kY \to S^kY$ be the natural projections. Let $L_j = \pi_1^*\mathcal{L}_j$. Consider the exact sequence of sheaves on $Y \times S^kY$

$$0 \to L_j \to L_j \otimes \mathcal{O}_{Y \times S^kY}(\Delta_Y) \to \tau_*(\pi_Y^*\mathcal{L}_j \otimes \mathcal{O}_{\Delta_Y}(\Delta_Y)) \to 0, \tag{2}$$

where $\tau$ stands for the embedding $\Delta_Y \hookrightarrow Y \times S^kY$. The higher direct images $R^i\pi_{S^kY*}(\pi_Y^*\mathcal{L}_j \otimes \mathcal{O}_{\Delta_Y}(\Delta_Y))$ vanish for $i \geq 1$, so using the Grothendieck–Riemann–Roch theorem we have

$$\begin{aligned} &td(S^kY)ch(i^*T_{S^{pk}X}(\nu^j)) \\ &\quad= (\pi_2)_*(td(Y \times S^kY) \cdot ch(L_j \otimes \tau_*\mathcal{O}_{\Delta_Y}(\Delta_Y))) \\ &\quad= (\pi_2)_*(\pi_1^*(td(Y)) \cdot \pi_2^*(td(S^kY)) \cdot ch(L_j \otimes \tau_*\mathcal{O}_{\Delta_Y}(\Delta_Y))). \end{aligned}$$

Cancelling out $Td(S^kY)$ and using the exact sequence (2) to compute $ch(L_j \otimes \tau_* \mathcal{O}_{\Delta_Y}(\Delta_Y))$, we get

$$ch(i^* T_{S^{pk}X}(\nu^j)) = (\pi_2)_*((e^{n_j\eta+\delta} - e^{n_j\eta})(1 + (1 - g_Y)\eta),$$

where $\delta$ is the class of $\Delta_Y$ in $H^2(Y \times S^kY, \mathbb{Z})$ and $\eta$ is the pull-back under $\pi_1$ of the class of a point in $Y$. Let $x' = \pi_2^*(x)$, $\theta' = \pi_2^*(\theta) \in H^2(Y \times S^kY, \mathbb{Z})$. One can write $\delta = x' + \gamma + k\eta$; under the Kunnet decomposition, $\gamma$ is the component $\delta^{1,1}$ of $\delta$. One has $\gamma^2 = -2\eta\theta'$, $\eta^2 = \eta\gamma = \gamma^3 = 0$; see [1], chapter VIII, section 2, p. 338. Using

$$e^\delta = e^{k\eta+\gamma+x'} = e^{x'} + k\eta e^{x'} - \eta\theta' e^{x'} + \gamma e^{x'},$$

one gets

$$\begin{aligned} e^{n_j\eta+\delta} - e^{n_j\eta} &= (1 + n_j\eta)(e^{x'} \cdot (1 + k\eta - \eta\theta' + \gamma) - 1) \\ &= e^{x'} \cdot (1 + \eta(k - \theta' + n_j) + \gamma) - (1 + n_j\eta); \end{aligned}$$

then

$$\begin{aligned} &(e^{n_j\eta+\delta} - e^{n_j\eta})(1 + (1 - g)\eta) = \\ &\qquad e^{x'} \cdot (1 + \gamma + \eta(k - \theta' + n_j + 1 - g_Y)) - 1 - (n_j + 1 - g_Y)\eta. \end{aligned}$$

Notice $(\pi_2)_*(\delta) = k$, $(\pi_2)_*(\eta) = 1$, $(\pi_2)_*(1) = (\pi_2)_*(x') = 0$; then $(\pi_2)_*(\gamma) = 0$ and

$$\begin{aligned} ch(i^* T_{S^{pk}X}(\nu^j)) &= e^x \cdot (\pi_2)_*(1 + \gamma + \eta(k - \theta' + n_j + 1 - g_Y)) - (n_j + 1 - g_Y) \\ &= g_Y - n_j - 1 + (k + n_j + 1 - g_Y - \theta)e^x. \end{aligned}$$

Now $ch(i^* T_{S^{pk}X}(\nu^j)) + Be^x$ is the Chern character of something that has Chern class $(1+x)^B c(i^* T_{S^{pk}X}(\nu^j))$. So the total Chern class can be deduced from the following observation. If $(r - \theta)e^x$ is the Chern character of a rank $r$ vector bundle $E$ on $S^kY$, then $c(E) = (1+x)^r e^{-\theta/(1+x)}$. To see this, notice that $r - \theta$ can be seen as the Chern character of a rank $r$ vector bundle $F$ with $c(F) = e^{-\theta}$ (one can assume that $r \geq g_Y$ and that the non-zero Chern roots of $F$ are $-\sigma_1, \ldots, -\sigma_{g_Y}$), $e^x$ is the Chern character of a line bundle $L$ with $c(L) = 1 + x$, so $ch(F \otimes L) = (r - \theta)e^x$. Then $c(F \otimes L) = (1+x)^{r-g_Y} \prod_{i=1}^{g_Y}(1 - \sigma_i + x) = (1+x)^r e^{-\theta/(1+x)}$. ∎

Let $N_i(\nu^j)$ have Chern roots $\{x_s\}$; then the corresponding stable characteristic class is given by

$$\mathcal{U}_j(N_i(\nu^j)) = \prod_s \left(\frac{1 - e^{-x_s}/\nu^j}{1 - \nu^{-j}}\right)^{-1}.$$

Notice from the definition of $\mathcal{U}_j$ that $\mathcal{U}_j(E)$ depends only on the Chern classes $c_i(E)$ of $E$ and not on the $x_s$'s, that is, $\mathcal{U}_j$ is a transformation of polynomials and it satisfies $\mathcal{U}_j(p_1p_2) = \mathcal{U}_j(p_1)\mathcal{U}_j(p_2)$.

As a corollary we have the following

THEOREM 2.3: *Let $n_j$ be the degree of $\mathcal{L}_j$. Then*

$$\text{(a)}\quad \mathcal{U}_j(N_i(\nu^j)) = (1-1/\nu^j)^A(1-e^{-x}/\nu^j)^{-A}e^{\theta(\frac{e^{-x}}{\nu^j-e^{-x}})}\left(\frac{1-e^{-x}/\nu^j}{1-\nu^{-j}}\right)^{-n_j},$$

$$\text{(b)}\quad \prod_{j=1}^{p-1}\mathcal{U}_j(N_i(\nu^j)) = p^A m(e^{-x})^{-A}e^{\theta q(e^{-x})}\prod_{j=1}^{p-1}\left(\frac{1-e^{-x}/\nu^j}{1-\nu^{-j}}\right)^{-n_j},$$

*where $A = k+1-g_Y$, $m(z) = \sum_{i=0}^{p-1} z^i$, $q(z) = -zm'(z)/m(z)$.*

*Proof:* Using equation (1), one can use the proof of Lemma 3.7 in [4]. ∎

*Remark:* Notice that in the case $k = 1$ of Proposition 2.1 one has $i^*T_{S^pX} \cong f_*f^*K_Y^{-1}$ which, by the relative Serre duality, implies that $i^*\Omega^1_{S^pX} \cong f_*K_X$. Then one can identify $H^0(Y,\mathcal{L}_j^{-1}K_Y)$ with the eigenspace $H^0(X,K_X)(\nu^{-j})$ and, using Riemann–Roch theorem, one can compute the degree of $\mathcal{L}_j^{-1}K_Y$. The decomposition of $H^0(X,K_X)$ into eigenspaces of $h$ can be obtained by applying the Atiyah–Bott fixed point theorem.

Proposition 2.1 also can be used to compute $c(i^*(K_{S^{pk}X}))$. Namely,

$$c_1(i^*T_{S^{pk}X}) = \sum_{j=0}^{p-1}(k+n_j+1-g_Y)x - \theta = (pk+1-g_X)x - p\theta.$$

## References


[1] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of Algebraic Curves*, Springer-Verlag, New York, 1985.

[2] M. F. Atiyah and I. M. Singer, *The index of elliptic operators III*, Annals of Mathematics **87** (1968), 547–604.

[3] I. G. Macdonald, *Symmetric products of an algebraic curve*, Topology **1** (1962), 319–343.

[4] I. Moreno Mejía, *Characteristic classes on symmetric products of curves*, Glasgow Mathematical Journal **46** (2004), 477–488.

[5] I. Moreno Mejía, *The trace of an automorphism on* $H^0(J,\mathcal{O}(n\Theta))$, Michigan Mathematical Journal **53** (2005), 57–69.